The Fekete-$Szeg\ddot{o}$ Functional and Second Hankel Determinant for a Subclass of Univalent Functions related to complex order

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Abstract

Let $\mathcal{A}$ be the class of analytic, univalent and normalized functions $f(z)$ with $f(z) = z + \sum_{j=2}^{\infty} c_j z^j$ for $z \in \triangle := \{z: |z| < 1\}$. The generalized hypergeometric functions and the Dziok-Srivastava operator $\psi_n^m[\theta]f(z)$ are utilized as follows;

$\psi_n^m[\theta]f(z) = z + \sum_{j=2}^{\infty} \Gamma_j c_j z^j$, where $\Gamma_j = \frac{(\theta_1)_{j-1},\dots,(\theta_m)_{j-1}}{(\eta_1)_{j-1},\dots,(\eta_n)_{j-1}} \frac{1}{(j-1)!}$ to introduce the following subclass. a new subclass of complex order $R_n^m(\theta, \eta, \delta)$ of $\mathcal{A}$ is introduced as follows;

$$Re\left\{1 + \frac{1}{\delta}\left(\frac{z(\psi_n^m[\theta]f(z))'}{\psi_n^m[\theta]f(z)} - 1\right)\right\} > 0, \qquad (\delta \in \mathbb{C} - \{0\}, z \in \triangle).$$

Estimated coefficients, the Fekete-$Szeg\ddot{o}$ functional and the Hankel determinant are obtained for this new subclass $R_n^m(\theta, \eta, \delta)$.

Key words: Analytic and Univalent functions, Estimated coefficients, Dziok-Srivastava operator, Fekete-$Szeg\ddot{o}$ functional, Hankel Determinant.

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1. Introduction

We consider $\mathcal{A}$ to be the class of regular and univalent functions

$$f(z) = z + \sum_{j=2}^{\infty} c_j z^j, \qquad (1.1)$$

with $z \in \triangle := \{z: |z| < 1\}$ and the conditions $f(0) = 0$ and $f'(0) = 1$ are satisfied.

"The generalized hypergeometric function" ${}_mL_n(z)$ is defined for positive real numbers $\theta_1, \dots, \theta_m$ and $\eta_1, \dots, \eta_n$ $(\eta_s \neq 0, -1, \dots; s = 1,2,\dots,n)$, by

$$ {}_mL_n(z) = {}_mL_n(\theta_1, \dots, \theta_m; \eta_1, \dots, \eta_n; z) = \sum_{j=0}^{\infty} \frac{(\theta_1)_j \cdots (\theta_m)_j}{(\eta_1)_j \cdots (\eta_n)_j} \frac{z^j}{j!} \qquad (1.2)$$

$(m \leq n + 1;\ m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ z \in \triangle)$, where as $(\tau)_j$ is the "Pochhammer symbol" given with

$$(\tau)_j = \begin{cases} 1, & j = 0 \\ \tau(\tau + 1) \dots (\tau + j - 1), & j \in \mathbb{N} \end{cases}$$

"Many well-known functions as the exponential, the Binomial, the Bessel and others" are represented by ${}_mL_n$ (c.f [3]).

The linear operator $\psi(\theta_1, \dots, \theta_m; \eta_1, \dots, \eta_n): \mathcal{A} \to \mathcal{A}$ called the Dziok-Srivastava operator (c.f [3]) is defined as follows;

$$\psi(\theta_1, \dots, \theta_m; \eta_1, \dots, \eta_n)f(z) = z\ {}_mL_n(\theta_1, \theta_2, \dots, \theta_m; \eta_1, \eta_2, \dots, \eta_n; z) * f(z) = z + \sum_{j=2}^{\infty} \mathbb{Q}_j c_j z^j, \qquad (1.3)$$

where

$$\mathbb{Q}_j = \frac{(\theta_1)_{j-1},\dots,(\theta_m)_{j-1}}{(\eta_1)_{j-1},\dots,(\eta_m)_{j-1}} \frac{1}{(j-1)!}.$$

For each $\delta \in \mathbb{C} - \{0\}$, we introduce the subclass $R_n^m(\theta, \eta, \delta)$ of functions $f(z)$ of the form (1.1) with the following condition

$$Re\left\{1 + \frac{1}{\delta}\left(\frac{z(\psi_n^m[\theta]f(z))'}{\psi_n^m[\theta]f(z)} - 1\right)\right\} > 0, \quad z \in \triangle \ (1.6)$$

where $\psi_n^m[\theta]f(z)$ is defined by (1.3).

The following subclasses are obtained by giving certain values for m, n, θ_m and η_n for m ≤ n + 1;

- For n = 1, m = 2, $\theta_1 = 1$ and $\theta_2 = \eta_1$, we get the subclass that was obtained by Nasr and Aouf [10].
- For n = 1, m = 2, $\theta_1 = 2$ and $\theta_2 = \eta_1$, we get the subclass that was obtained by Nasr and Aouf [9].

The estimate of $|c_3 - \varrho c_2^2|$ is investigated by Fekete and $Szeg\ddot{o}$ [4] for a function $f(z)$ with a Salagean differential operator, called as "Fekete and $Szeg\ddot{o}$ functional", where ϱ is real. Also, the q^{th} Hankel determinant is obtained by Noonan and Thomas [11] for $f(z) \in \mathcal{A}$ for $q \geq 1$ and $i \geq 0$ as follows

$$H_q(i) = \begin{vmatrix} c_i & c_{i+1} & \cdots & c_{i+q-1} \\ c_{i+1} & c_{i+2} & \cdots & c_{i+q} \\ \vdots & \vdots & \ddots & \vdots \\ c_{i+q-1} & c_{i+q} & \cdots & c_{i+2q-2} \end{vmatrix}$$

$(c_1 = 1)$.

This determinant was obtained by various authors for certain values q and i. $H_2(1) = |c_3 - c_2^2|$ is called the Fekete-$Szeg\ddot{o}$ functional. "The Hankel determinant of p-valent functions, univalent functions and starlike functions" was obtained by Pommerenke [13]. Also Noor [12] studied "the Hankel determinant problem for the class of functions wit boundary rotation". The maximization value for "The second Hankel determinant" $H_2(2) = |c_2c_4 - c_3^2|$ for "univalent functions whose derivative has positive real parts" was studied by Janteng et al. [5]. Also, Lee et al. [7] investigated the maximization value for "the second Hankel determiant for subclasses of Ma-Minda starlike and convex functions". Bansal [1] obtained $H_2(2)$ for a new subclass of regular functions. For more see [13,14]

We seek in our investigation to obtain the estimated coefficients and "the second Hankel determinant" for the functions $f(z)$ which belongs to $R_n^m(\theta,\eta,\delta)$.

2. Initial Lemmas

The following lemmas that we need to obtain our results are mentioned. Suppose ω be the class of functions of the form

$v(z) = 1 + b_1z + b_2z^2 + b_3z^3 + \ldots, \quad (z \in \triangle)$

that are regular with the conditions $v(0) = 1$ and $Re\{v(z)\} > 0$.

Lemma 1 [2]

Suppose $v(z) \in \omega$, then

$$|b_t| \leq 2 \quad (t \in \mathbb{N}). \tag{2.1}$$

Lemma 2 [6]

Suppose $v(z) \in \omega$, then

$$|b_2 - tb_1^2| \leq 2\max\{1, |2t - 1|\} \tag{2.2}$$

for any complex number t.

Lemma 3 [8]

Suppose $v(z) \in \omega$, then we have for certain values of x and z

$$2b_2 = b_1^2 + x(4b_1^2), \tag{2.3}$$

$$4b_3 = b_1^3 + 2b_1(4 - b_1^2)x - c_1(4 - b_1^2)x^2 + 2(4 - b_1^2)(1 - |x|^2)z \tag{2.4}$$

where as $|x| \leq 1$ and $|z| \leq 1$.

3. Main Results

Theorem 1.

Let $f(z) \in R_n^m(\theta,\eta,\delta)$, then

$$|c_2| \leq \frac{2|\delta|}{\mathbb{Q}_2},$$

$$|c_3| \leq \frac{|\delta|(1 + 2|\delta|)}{\mathbb{Q}_3}.$$

Proof. Let $f(z) \in R_n^m(\theta,\eta,\delta)$, thus we obtain by (1.6)

$$Re\{1 + \frac{1}{\delta}[\frac{1+\sum_{j=2}^{\infty} j\mathbb{Q}_jc_jz^{j-1}}{1+\sum_{j=2}^{\infty}\mathbb{Q}_jc_jz^{j-1}} - 1]\} > 0, \tag{3.1}$$

where $\mathbb{Q}_j = \frac{(\theta_1)_{j-1},\ldots,(\theta_m)_{j-1}}{(\eta_1)_{j-1},\ldots,(\eta_n)_{j-1}}\frac{1}{(j-1)!}$.

By setting

$$1 + \frac{1}{\delta}\left[\frac{1 + \sum_{j=2}^{\infty} j\mathbb{Q}_jc_jz^{j-1}}{1 + \sum_{j=2}^{\infty}\mathbb{Q}_jc_jz^{j-1}} - 1\right] = v(z) = 1 + b_1z + b_2z^2 + b_3z^3 + \ldots, \tag{3.2}$$

then we obtain

$$1 + \frac{1}{\delta}\left[\frac{1+\sum_{j=2}^{\infty} j\mathbb{Q}_jc_jz^{j-1}}{1+\sum_{j=2}^{\infty}\mathbb{Q}_jc_jz^{j-1}} - 1\right] = 1 + \frac{1}{\delta}\Big[\big(1 + \sum_{j=2}^{\infty} j\mathbb{Q}_jc_jz^{j-1}\big) * (1 - [\mathbb{Q}_2c_2z + \mathbb{Q}_3c_3z^2]) + \mathbb{Q}_4c_4z^3 + \cdots] + [\mathbb{Q}_2c_2z + \mathbb{Q}_3c_3z^2 + \mathbb{Q}_4c_4z^3 + \cdots]^2 - [\mathbb{Q}_2c_2z + \mathbb{Q}_3c_3z^2 + \mathbb{Q}_4c_4z^3 + \cdots]^3 + \cdots) - 1\Big]$$

$$= 1 + b_1z + b_2z^2 + b_3z^3 + \ldots \; from \; (3.2). \tag{3.3}$$

Thus by the comparison of the coefficients in (3.3), we obtain

$$c_2 = \frac{\delta b_1}{\mathbb{Q}_2}, \tag{3.4}$$

$$c_3 = \frac{\delta(b_2 + \delta b_1^2)}{2\mathbb{Q}_3}, \tag{3.5}$$

$$c_4 = \frac{\delta[2b_3 + \delta^2b_1^3 + 3\delta b_1b_2]}{3!\mathbb{Q}_4}. \tag{3.6}$$

We obtain by using lemma 1 with (3.4) and (3.5), (3.6)

$$|c_2| \leq \frac{2|\delta|}{\mathbb{Q}_2} \tag{3.7}$$

$$|c_3| \leq \frac{|\delta|(1 + 2|\delta|)}{\mathbb{Q}_3} \tag{3.8}$$

$$|c_4| \leq \frac{2|\delta|[1+2|\delta|^2+3|\delta|]}{3\mathbb{Q}_4}. \tag{3.9}$$

Thus the proof is complete.

Theorem 2.

Suppose $f(z) \in R_n^m(\theta,\eta,\delta)$, then we have for any real number γ

$$|c_3 - \gamma c_2^2| \leq \frac{|\delta|}{\mathbb{Q}_3}\max\left\{1, \left|2\delta\left\{\frac{2\gamma\mathbb{Q}_3}{\mathbb{Q}_2^2} - 1\right\} - 1\right|\right\}. \tag{3.10}$$

Proof. Using (3.4) and (3.5), we obtain

$$|c_3 - \gamma c_2^2| = \frac{|\delta|}{2\mathbb{Q}_3}\left|b_2 - \delta\left\{\frac{2\gamma\mathbb{Q}_3}{\mathbb{Q}_2^2} - 1\right\}b_1^2\right|. \tag{3.11}$$

Upon applying Lemma 2, we have

$$|c_3 - \gamma c_2^2| \leq \frac{|\delta|}{\mathbb{Q}_3}\max\left\{1, \left|2\delta\left\{\frac{2\gamma\mathbb{Q}_3}{\mathbb{Q}_2^2} - 1\right\} - 1\right|\right\}. \tag{3.12}$$

Thus we end the proof.

Theorem 3 .

Let $f(z) \in R_n^m(\theta,\eta,\delta)$, then

$$H_2(2) = |c_2c_4 - c_3^2| \leq \frac{|\delta|^2}{3\mathbb{Q}_2\mathbb{Q}_4\mathbb{Q}_3^2}\left\{4\mathbb{Q}_3^2\big(1 + |\delta|(2|\delta| + 3)\big) + 3\mathbb{Q}_2\mathbb{Q}_4\big(1 + 4|\delta|(1 + |\delta|)\big)\right\}. \tag{3.13}$$

Proof. We get from (3.4), (3.5) and (3.6)

$$|c_2c_4 - c_3^2| = \frac{|\delta|^2}{12\mathbb{Q}_2\mathbb{Q}_4\mathbb{Q}_3^2}|\mathbb{Q}_3^2(4b_1b_3 + 2\delta^2b_1^4 + 6\delta b_1^2b_2) - 3\mathbb{Q}_2\mathbb{Q}_4(b_2 + \delta b_1^2)^2|. \tag{3.14}$$

Upon applying Lemma 3, we get

$|c_2c_4 - c_3^2| = \frac{|\delta|^2}{12\mathbb{Q}_2\mathbb{Q}_4\mathbb{Q}_3^2}|\mathbb{Q}_3^2(b_1^4 + 2b_1^2(4 -$

$b_1^2)x - b_1^2(4-b_1^2)x^2 + 2b_1(4-b_1^2)(1-|x|^2)z + 2\delta^2 b_1^4 + 3\delta b_1^4 + 3\delta b_1^2(4-b_1^2)x) - \frac{3}{4}\mathbb{Q}_2\mathbb{Q}_4(b_1^2 + (4-b_1^2)x + 2\delta b_1^2)^2$.

Setting $b_1 = b$ and since $|b_1| \le 2$, then free of limitations we assume $b \in [0,2]$.

$$|c_2c_4 - c_3^2| \le \frac{|\delta|^2}{12\mathbb{Q}_2\mathbb{Q}_4\mathbb{Q}_3^2}\{\mathbb{Q}_3^2(b^4 + 2b^2(4-b^2)|x| + b^2(4-b^2)|x|^2 + 2b(4-b^2)(1-|x|^2) + 2|\delta|^2 b^4 + 3|\delta| b^4 + 3|\delta| b^2(4-b^2)|x|) + \frac{3}{4}\mathbb{Q}_2\mathbb{Q}_4(b^4 + (4-b^2)^2|x|^2 + 2b^2(4-b^2)|x| + 4|\delta|^2 b^4 + 4|\delta| b^4 + 4|\delta| b^2(4-b^2)|x|).$$

Setting $\mathcal{D} = |x| \le 1$, we get

$$|c_2c_4 - c_3^2| \le \frac{|\delta|^2}{12\mathbb{Q}_2\mathbb{Q}_4\mathbb{Q}_3^2}[b^4\mathbb{Q}_3^2 + 2b(4-b^2)\mathbb{Q}_3^2 + 2|\delta|^2\mathbb{Q}_3^2 b^4 + 3|\delta|\mathbb{Q}_3^2 b^4 + \frac{3}{4}\mathbb{Q}_2\mathbb{Q}_4 b^4 + 3|\delta|^2\mathbb{Q}_2\mathbb{Q}_4 b^4 + 3|\delta|\mathbb{Q}_2\mathbb{Q}_4 b^4 + \{2\mathbb{Q}_3^2 b^2(4-b^2) + 3|\delta|\mathbb{Q}_3^2 b^2(4-b^2) + \frac{3}{2}\mathbb{Q}_2\mathbb{Q}_4 b^2(4-b^2) + 3|\delta|\mathbb{Q}_2\mathbb{Q}_4 b^2(4-b^2)\}\mathcal{D} + \{b^2(4-b^2)\mathbb{Q}_3^2 - 2b(4-b^2)\mathbb{Q}_3^2 + \frac{3}{4}\mathbb{Q}_2\mathbb{Q}_4(4-b^2)^2\}\mathcal{D}^2] = T(b,\mathcal{D}). \quad (3.15)$$

Thus for $0 \le \mathcal{D} \le 1$

$$2\mathbb{Q}_3^2 b^2(4-b^2) + 3|\delta|\mathbb{Q}_3^2 b^2(4-b^2) + \frac{3}{2}\mathbb{Q}_2\mathbb{Q}_4 b^2(4-b^2) + 3|\delta|\mathbb{Q}_2\mathbb{Q}_4 b^2(4-b^2) + \{2b^2(4-b^2)\Gamma_3^2 - 4b(4-b^2)\mathbb{Q}_3^2 + \frac{3}{2}\mathbb{Q}_2\mathbb{Q}_4(4-b^2)^2\}\mathcal{D} > 0.$$

We note that $T(b,\mathcal{D})$ obtains its maximum value at $\mathcal{D} = 1$. Then

$$\max_{0\le\mathcal{D}\le 1} T(b,\mathcal{D}) = T(b,1) = \frac{|\delta|^2}{12\mathbb{Q}_2\mathbb{Q}_4\mathbb{Q}_3^2}[b^4\mathbb{Q}_3^2 + 2b(4-b^2)\mathbb{Q}_3^2 + 2|\delta|^2\mathbb{Q}_3^2 b^4 + 3|\delta|\mathbb{Q}_3^2 b^4 + \frac{3}{4}\mathbb{Q}_2\mathbb{Q}_4 b^4 + 3|\delta|^2\mathbb{Q}_2\mathbb{Q}_4 b^4 + 3|\delta|\mathbb{Q}_2\mathbb{Q}_4 b^4 + 2\mathbb{Q}_3^2 b^2(4-b^2) + 3|\delta|\mathbb{Q}_3^2 b^2(4-b^2) + \frac{3}{2}\mathbb{Q}_2\mathbb{Q}_4 b^2(4-b^2) + 3|\delta|\mathbb{Q}_2\mathbb{Q}_4 b^2(4-b^2) + c^2(4-c^2)\Gamma_3^2 - 2b(4-b^2)\mathbb{Q}_3^2 + \frac{3}{4}\mathbb{Q}_2\mathbb{Q}_4(4-b^2)^2] = \mathcal{W}(b). \quad (3.16)$$

$$\mathcal{W}'(b) = \frac{|\delta|^2}{3\mathbb{Q}_2\mathbb{Q}_4\mathbb{Q}_3^2}[(3|\delta|^2\mathbb{Q}_2\mathbb{Q}_4 + 2|\delta|^2\mathbb{Q}_3^2 - 2\mathbb{Q}_3^2)b^3 + (6\mathbb{Q}_3^2 + 6|\delta|\mathbb{Q}_3^2 + 6|\delta|\mathbb{Q}_2\mathbb{Q}_4)b] > 0, \quad (3.17)$$

for $0 \le c \le 2$. Since the function $\mathcal{W}(b)$ obtains its maximum value at $b = 2$, then

$$\mathcal{W}(b) = \frac{|\delta|^2}{3\mathbb{Q}_2\mathbb{Q}_4\mathbb{Q}_3^2}\{4\mathbb{Q}_3^2(1 + |\delta|(2|\delta| + 3)) + 3\mathbb{Q}_2\mathbb{Q}_4(1 + 4|\delta|(1 + |\delta|))\}. \quad (3.18)$$

(3.19)

Thus

$$|c_2c_4 - c_3^2| \le \frac{|\delta|^2}{3\mathbb{Q}_2\mathbb{Q}_4\mathbb{Q}_3^2}\{4\mathbb{Q}_3^2(1 + |\delta|(2|\delta| + 3)) + 3\mathbb{Q}_2\mathbb{Q}_4(1 + 4|\delta|(1 + |\delta|))\}. \quad (3.19)$$

References

[1] D. Bansal, Upper bound of second Hankel determinant for a new class of analytic functions, J. Appl. Math. Lett., 26 (1) (2013), 103-107.

[2] P. L. Duren, Univalent Functions, Springer-Verlag New York, 259 (1) (1983).

[3] J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Intrgral Transforms and Special Functions, 14 (1), 2003, 7-18.

[4] M. Fekete and G. *Szegö*, Eine Bemerkung *Über* Ungerade Schlichte Funktionen, J. london Math. Soc., 1 (2) (1933), 85-89.

[5] A. Janteng, S. A. Halim and M. Darus, Hankel determinant for starlike and convex functions, Int. J. Math. Anal., 1 (13) (2007), 619-625.

[6] F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc., 20 (1969), 8-12.

[7] S. K. Lee, V. Ravichandran and S. Supramaniam, Bounds for the second Hankel

determinant of certain univalent functions, J. Inequal. Appl., (1) (2013), 1-17.

[8] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative, Proc. Am. Math. Soc. 87 (1983), 251-257.

[9] M. A. Nasr and M. K. Aouf, On convex functions of complex order, Mansoura Sci. Bull. Egypt 9 (1982), 565-582.

[10] M. A. Nasr and M. K. Aouf, Starlike Functions of complex order, J. Nature. Sci. Math., 25 (1) (1985), 1-12.

[11] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of a really mean p-valent functions, Trans. Am. Math. Soc, 223 (1976), 337-346.

[12] K. I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, Revue Roumaine de Mathématique Pures et Appliquées, 28 (1983), 731-739.

[13] C. Pommerenke, On the coefficients and Hankel determinant of univalent functions, J. London Math. Soc., 1(1) (1966), 111-122.

[14] J. K. Prajapat, D. Bansal, A. Singh and A. K. Mishra, Bounds on third Hankel determinant for close -to- convex functions, Acta Universitatis Sapientiae, Mathematica, 7(2) (2015), 210-219.

[15] J. Sok$\acute{o}l$ and D. K. Thomas, The second Hankel determinant for alpha - convex functions, Lithuanian Mathematical Journal, 58 (2) (2018), 212-218.